AMERICAN Journal of Science on Integration and Human Development
Volume 3, Issue 12, 2025 ISSN (E): 2993-2750

On Dynamics of a Volterra Cubic Stochastic Operators in S^2

Mahmatqobilov Nurbek Po'lat o'g'li
Faculty of Mathematics, Karshi State University, Karshi, Uzbekistan, Teacher

Zikrmuhammad Abdusamadov
University of Information Technologies and Management, Karshi, Uzbekistan, Master

Abstract: In this article, we study the Volterra cubic stochastic operator defined on a two-dimensional simplex. That is, we consider the cubic stochastic operator defined on a simplex. It is shown that for the Volterra cubic stochastic operator defined on a two-dimensional simplex, the vertices of the simplex are fixed points.

Keywords: Cubic stochastic operator; volterra cubic stochastic operator; orbit, simplex.

INTRODUCTION

There are many systems which are described by nonlinear operators. A quadratic stochastic operator (QSO) is one of the simplest nonlinear cases. A QSO has meaning of a population evolution operator and it was first introduced by Bernstein in [1]. For more than 80 years, the theory of QSOs has been developed and many papers were published (see e.g. [4]-[8], [16]-[17]). In recent years it has again become of interest in connection with its numerous applications in many branches of mathematics, biology and physics.

Let $E=\{1,2,...,m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1}=\left\{x=(x_1,\dots x_m)\in\mathbb{R}^m : x_i\geq 0,\sum_{i=1}^{m}x_i=1\right\} \quad (1)$$

be the $(m-1)$-dimensional simplex. A QSO is a mapping defined as $V:S^{m-1}\to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x})=\mathbf{x}'\in S^{m-1}$, where

$$x'_k=\sum_{i,j=1}^{m}P_{ij,k}x_ix_j,\qquad k\in E, \quad (2)$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k}=P_{ji,k}\geq 0,\quad \sum_{k=1}^{m}P_{ij,k}=1 \text{ for all } i,j\in E. \quad (3)$$

The trajectory (orbit) $\{\mathbf{x}^{(n)}\}_{n\geq 0}$, of V for an initial value $\mathbf{x}^{(0)}\in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V\left(\mathbf{x}^{(n)}\right) = V^{(n+1)}\left(\mathbf{x}^{(0)}\right), \qquad n = 0,1,2,\dots .$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

The operator V is called Volterra QSO, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k \in E$. For the Volterra QSO the general formula was given in [4],

$$x_k' = x_k\left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad (4)$$

where $a_{ki} = 2P_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$ for all $i, k \in E$.

In [4], the theory of Volterra QSO was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs were not completely studied. Because, there is no general theory that can be applied for study of non-Volterra operators.

In recent years, Cubic Stochastic Operators (CSOs) have begun to be studied, which different from quadratic operators [8-10].

RESULTS AND DISCUSSION

Definition 1. [16] *The QSO (6) is called separable quadratic stochastic operator (SQSO).*

Cubic stochastic operator. The CSO is a mapping $W : S^{m-1} \to S^{m-1}$ of the form

$$x_l' = \sum_{i,j,k=1}^{m} P_{ijk,l} x_i x_j x_k, \quad l \in E, \quad (5)$$

where $P_{ijk,l}$ are coefficients of heredity such that

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} P_{ijk,l} = 1, \quad \forall i, j, k \in E. \quad (6)$$

and we suppose that the coefficients $P_{ijk,l}$ do not change for any permutation of i, j, k.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n\geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of CSO (8) is defined by $\mathbf{x}^{(n+1)} = W\left(\mathbf{x}^{(n)}\right)$, where $n = 0,1,2,\dots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega\left(\mathbf{x}^{(0)}\right)$ the set of limit points of the trajectory $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$. Since $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it is follows that $\omega\left(\mathbf{x}^{(0)}\right) \neq \varnothing$. If $\omega\left(\mathbf{x}^{(0)}\right)$ consists of a single point, they the trajectory converges and $\omega\left(\mathbf{x}^{(0)}\right)$ is a fixed point of the operator W. A point $\mathbf{x} \in S^{m-1}$ is called a fixed of the W if $W(\mathbf{x}) = \mathbf{x}$. Denote by $\mathrm{Fix}(W)$ the set of all fixed points of the operator W, i.e.

$$\mathrm{Fix}(W) = \left\{\mathbf{x} \in S^{m-1} : W(\mathbf{x}) = \mathbf{x}\right\}.$$

The Volterra CSO $W : S^2 \to S^2$ is:

$$W:\begin{cases} x_1' = x_1(1+x_2)(1+x_3), \\ x_2' = x_2(1-x_3)\left(1+\frac{1}{2}x_1\right), \\ x_3' = x_3(1-x_1)\left(1-\frac{1}{2}x_2\right). \end{cases} \quad (7)$$

Let a face of the simplex S^2 be the set $\Gamma_\alpha = \{\mathbf{x} \in S^2 : x_i = 0,\, i \notin \alpha \subset \{1,2,3\}\}$.

Let the set $\operatorname{int} S^2 = \{\mathbf{x} \in S^2 : x_1x_2x_3 > 0\}$ and let the set $\partial S^2 = S^2 \setminus \operatorname{int} S^2$ be the interior and the boundary of the simplex S^2, respectively. Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ be the vertexes of the two-dimensional simplex.

Lemma. *For the SCSO* W (7), *the following assertions true*:

(i) *The face* $\Gamma_{\{1,2\}}$, $\Gamma_{\{1,3\}}$, $\Gamma_{\{2,3\}}$ *of the simplex* S^2 *are invariant sets;*

(ii) $\operatorname{Fix}(W) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$*;*

CONCLUSION.

Currently, the theory of nonlinear dynamical systems is used as the main tool in solving many practical problems of mathematical biology worldwide. In particular, the problem of determining the dynamics of separable cubic stochastic operators, due to the nonlinearity of these operators, raises many theoretical and practical issues. In this regard, targeted scientific research is focused on describing the invariant sets of such operators, finding periodic points and determining their types, and describing the set of limit points of orbits.

REFERENCES

1. Bernstein S. N. Solution of a mathematical problem connected with the theory of heredity, // *The Annals of Mathematical Statistics.* pp. 83 (1942).

2. Davronov R.R., Jamilov U.U and M. Ladra. Conditional cubic stochastic operator, // *Difference Equ. Appl.* Vol **21**, Issue 12, pp. 1163–1170, (2015).

3. Devaney R.L. An introduction to chaotic dynamical systems, // *Boulder, Stud. Nonlinearity, Westview Press*, 2003.

4. Ganikhodzhaev R.N. Quadratic stochastic operators, Lyapunov functions and tournaments, // *Acad. Sci. Sb. Math.* Vol **76**, (1993), pp. 489-506.

5. Ganikhodzhaev R.N., Eshmamatova D.B. Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories, // *Vladikavkaz. Math. J.* Vol **8**, Issue 2, pp. 12–28, (2006).

6. Ganikhodzhaev R.N., Mukhamedov F.M., Rozikov U.A. Quadratic stochastic operators and processes: results and open problems, // *Inf. Dim. Anal. Quant.Prob. Rel. Fields* Vol **14**, Issue 2, pp. 279-335, (2011).

7. Jamilov U. U. On symmetric strictly non-Volterra quadratic stochastic operators, // Discontin. Nonlinearity Complex. Vol 5. Issue 3, pp. 263–283, (2016).
8. Jamilov U. U., Ladra M. On identically distributed non-Volterra cubic stochastic operator, // *J. Appl. Nonlinear Dyn*. Vol **6,** Issue 1, pp. 79–90, (2017).
9. Jamilov U. U., Reinfelds A. On constrained Volterra cubic stochastic operators, // *J. Differ. Equ. Appl*. Vol **26**. Issue 2, (2020) pp. 1-14.
10. Jamilov U. U., Khamraev A. Yu., Ladra M. On a Volterra cubic stochastic operator, // *Bull. Math. Biol*. Vol **80**. Issue 2, pp. 319-334, (2018).
11. Baratov, B.S., Jamilov, U.U. On separable cubic stochastic operators, // *Qual. Theory Dyn. Syst.* Vol **23**, Issue 2, a.n. 93. 28 pages. (2024).
12. Baratov B. S. On dynamics of a separable cubic stochastic operator, // *Bull. Inst. Math*., Vol **6**, Issue 1, pp. 17-25, (2023).
13. Baratov. B. S. The dynamics of a separable cubic operator, *Uz. Math. Jour,* Vol 68, Issue 2, pp. 27-41. 2024, DOI:10.29229/uzmj.2024-2-3.
14. Rozikov U. A., Khamraev A. Yu. On construction and a class of non-Volterra cubic stochastic operators, // *Nonlinear Dyn. Syst. Theory*, Vol **14**, Issue 1, pp.92–100, (2014).
15. Rozikov U. A., Khamraev A. Yu. On cubic operators defined on finite-dimensional simplexes, // Ukrainian Math. J. Vol 56, Issue 10, pp. 223-228, (2004).
16. Rozikov U.A., Nazir S. Separable quadratic stochastic operators, // *Lobachevskii J. Math.* Vol **31**, Issue 3, pp. 215-222, (2010).
17. Rozikov U.A., Zada A. On a class of separable quadratic stochastic operators, // *Lobachevskii Jour. Math.* Vol **32**, Issue 4, pp. 397-406, (2011).