AMERICAN Journal of Engineering, Mechanics and Architecture
Volume 01, Issue 07, 2023 ISSN (E): 2993-2637

Lie-Derivations of Some Nilpotent and Solvable Leibniz Algebras

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Abstract: Leibniz algebras are generalization of Lie algebras. These algebras preserve a unique property of Lie algebras that the right multiplication operators are derivations. The derivation operator on algebras and their generalizations are important object in non-associative algebras. We have a number of generalizations of derivations, one of which is Lie-derivation. In this work, we investigate Lie-derivations of solvable Leibniz algebras and describe Lie-derivations for some nilpotent Leibniz algebras and lie-derivations of naturally graded filiform Leibniz algebras. Moreover, we give the description of Lie-derivations for three-dimensional nilpotent Leibniz algebras. In addition, the following are defined here: Lie-derivations of solvable Leibniz algebras with filiform and nulfiliform nilradical.

Keywords: Leibniz algebras, Lie-derivation, filiform, nulfiliform, nilpotent, nilradical.

Leibniz algebras are generalization of Lie algebras. These algebras preserve a unique property of Lie algebras that the right multiplication operators are derivations. They first appeared in paper of A.M. Blokh in the 1960s, under the name of D-algebras, emphasizing their close relationship with derivations. The theory of D-algebras did not get as thorough an examination as it deserved immediately after its introduction. Later, the same algebras were introduced in 1993 by Jean-Louis Loday, who called them Leibniz algebras due to the identity they satisfy. The main motivation for the introduction of Leibniz algebras was to study the periodicity phenomena in algebraic K-theory. The derivation operator on algebras and their generalizations are important object in non-associative algebras. We have a number of generalizations of derivations, one of which is Lie-derivation. The notion of Lie-derivation was first given in 2019 in the work of G.R. Biogmam, J.M. Casas, and N. Pacheco Regos [3]. They studied Lie-central derivations, Lie-centroids, and Lie-stem Leibniz algebras.

Definition 1. A Leibniz algebra over the field K is a vector space L equipped with a bilinear map, called bracket, $[-,\ -]\ :\ L\times L\ \rightarrow L$ satisfying the Leibniz identity:

$$[x,\ [y,\ z]] = [[x,\ y],\ z] - [[x,\ z],\ y],$$

for all $x,\ y,\ z\in L$.

Definition 2. A linear map $d: L\rightarrow L$ of a Leibniz algebra $(L,\ [-,\ -])$ is said to be a derivation if for all $x,\ y\in L,$ the following condition holds:

$$d([x,\ y]) = [d(x),\ y] + [x,\ d(y)] .$$

For the Leibniz algebra $(L,\ [-,\ -])$ we define the product $[x,\ y]_{lie}=[x,\ y]+[y,x]$.Note that, this product is trivial if and only if L is a Lie algebra.

Definition 3. A linear map $D: L \to L$ of a Leibniz algebra $(L, [-, -])$ is said to be a Lie-derivation if for all $x, y \in L$, the following condition holds:

$$D([x, y]_{lie}) = [D(x), y]_{lie} + [x, D(y)]_{lie}$$

We denote by DerLie(L) the set of all Lie-derivations of a Leibniz algebra L, which can be equipped with a structure of Lie algebra by means of the usual bracket

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1, \text{ for all } D_1, D_2 \in Der(L).$$

Definition 4. Maximal nilindex for n - dimensional L Leibniz algebras is said null-filiform, if there exists [17]

$$\dim L^i = (n+1) - i, \qquad 1 \le i \le n+1$$

n - dimensional L Leibniz algebras is said filiform, if there exists

$$\dim L^i = n - i \ , \ 2 \le i \le n$$

It is known that there exist unique n-dimensional Leibniz algebra denoted by with the table of multiplication [3]:

$$NF_n : [e_i, e_1] = e_{i+1}, 1 \le i \le n-1.$$

Moreover, any naturally graded n-dimensional Leibniz algebra is isomorphic one of the following two algebras

$$\mathrm{F}_n^1 : [e_1, e_1] = e_3 \ , \ [e_i, e_1] = e_{i+1} \ , \ 2 \le i \le n-1.$$

$$\mathrm{F}_n^2 : [e_1, e_1] = e_3 \ , \ [e_i, e_1] = e_{i+1} \ , \ 3 \le i \le n-1.$$

In the following theorem we show that any Lie-derivation of the null-filiform and naturally graded filiform Leibniz algebras is a derivation.

Teorema 5. Any Lie-derivationof the algebras NF_n , $\mathrm{F}_n^1, \mathrm{F}_n^2$ is a derivation.

Now we consider three-dimensional nilpotent Leibniz algebras:

λ_1 : abelian ;

$\lambda_2 : [e_1, e_1] = e_2;$

$\lambda_3 : [e_2, e_3] = e_1, [e_3, e_2] = -e_1;$

$\lambda_4 : [e_2, e_2] = e_1, [e_3, e_2] = \alpha e_1, [e_2, e_3] = e_1 (\alpha \in C);$

$\lambda_5 : [e_2, e_2] = e_1, [e_3, e_2] = e_1, [e_2, e_3] = e_1;$

$\lambda_6 : [e_3, e_3] = e_1, [e_1, e_3] = e_2.$

Teorema 6. Any Lie-derivation of the algebras λ_1 , λ_2 , λ_3 , λ_5 , λ_6 is a derivation.

Proposition 7. There exists Lie-derivation of the algebra λ_4 which is not a derivation and any Lie-derivation of the algebra λ_4 has the following form:

$$D(e_1) = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3;$$

$$D(e_2) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3;$$

$$D(e_3) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3;$$

A Leibniz algebra L is said to be solvable, if there exists $n \in N$ $(m \in N)$ such that $L^{[m]} = 0$, where $L^{[1]} = L,\ L^{[k+1]} = \left[L^{[k]},\ L^{[k]}\right]$).

Consider following solvable Leibniz algebra R with basis $\{e_1, e_2, ..., e_n, x\}$ and table of multiplication is

$$R\begin{cases}[e_i,e_1]=e_{i+1}, 1\le i\le n-1\\ [x,e_1]=e_1\\ [e_i,x]=-e_i, 1\le i\le n\end{cases}$$

Proposition. Any Lie derivation of the solvable Leibniz algebra R is a derivation.

We consider solvable Leibniz algebras whose nilradical is non-split naturally graded filiform Leibniz algebras and codimension of the nilradical is equal to one.

$$R_1:\begin{cases}[e_i,e_1]=e_{i+1}, \quad 2\le i\le n-1\\ [x,e_1]=-e_1-e_2,\\ [e_1,x]=e_1,\\ [e_i,x]=(i-1)e_i; \ 2\le i\le n\end{cases}$$

$$R_2(\alpha):\begin{cases}[e_i,e_1]=e_{i+1}, \quad 2\le i\le n-1\\ [x,e_1]=-e_1,\\ [e_1,x]=e_1,\\ [e_i,x]=(i-1+\alpha)e_i; \ 2\le i\le n\end{cases}$$

$$R_3:\begin{cases}[e_i,e_1]=e_{i+1}, \quad 2\le i\le n-1\\ [x,e_1]=-e_1,\\ [e_1,x]=e_1,\\ [e_i,x]=(i-n)e_i \ ; \ 2\le i\le n\\ [x,x]=e_n\end{cases}$$

$$R_4:\begin{cases}[e_i,e_1]=e_{i+1}, \quad 2\le i\le n-1\\ [x,e_1]=-e_1,\\ [e_1,x]=e_1+e_n,\\ [e_i,x]=(i+1-n)e_i \ ; \ 2\le i\le n\\ [x,x]=-e_{n-1}.\end{cases}$$

Theorem. Any Lie-derivations of solvable Leibniz algebras $R_{1,}$, R_2, R_3, R_4 are derivations.

Proof. Let us first consider the derivation d of R_1 algebra. It is known that any derivation of the algebra R_1 has the following matrix form:

$$Der(R_1)=\begin{pmatrix}\alpha_1 & 0 & & & \\ & \beta_2 & & & \\ & & \alpha_1+\beta_2 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & (n-2)\alpha_1+\beta_2\end{pmatrix}$$

Then, we describe all Lie-derivation of the algebra R_1. Let D be a Lie-derivation of the algebra R_1. We set

$$D(e_1)=\sum_{i=1}^{n}\alpha_i e_i\ ,\quad D(e_2)=\sum_{i=1}^{n}\beta_i e_i\ ,\quad D(x)=\sum_{i=1}^{n}\gamma_i e_i$$

From the equality

$$0 = D\left([e_1, e_1]_{Lie}\right) = \left[D(e_1)e_1\right]_{Lie} + \left[e_1, D(e_1)\right]_{Lie} = D(e_1)e_1 + e_1D(e_1) + e_1D(e_1) +$$
$$+ D(e_1)\ e_1 = 2D(e_1)\ e_1 + 2e_1D(e_1) = 2(\alpha_1e_1 + \alpha_2e_2 + ... + \alpha_ne_n + \alpha_xx)e_1 +$$
$$+ 2e_1(\alpha_1e_1 + \alpha_2e_2 + ... + \alpha_ne_n + \alpha_xx) = 2\left(\alpha_2e_3+\alpha_3e_4+...+\alpha_{n-1}e_n+\alpha_x(-e_1-e_2) + \alpha_xe_1\right)=$$
$$= 2\left(-\alpha_xe_2+\alpha_2e_3+\alpha_3e_4+...+\alpha_{n-1}e_n\right) = 0$$

we get $\alpha_x = \alpha_2 = \alpha_3 = \ldots = \alpha_{n-1} = 0$.

Hence,

$$D(e_1) = \alpha_1 e_1 + \alpha_n e_n.$$

Further, we have

$$-D(e_2) = (\sum_{i=1}^{n} \gamma_i e_i)e_1 + e_1(\sum_{i=1}^{n} \gamma_i e_i) + x(\alpha_1 e_1 + \alpha_n e_n) + (\alpha_1 e_1 + \alpha_n e_n)x =$$

$$= \gamma_2 e_3 + \gamma_3 e_4 + \ldots + \gamma_{n-1} e_n + \gamma_x(-e_1 - e_2) + \gamma_x e_1 + \alpha_1(-e_1 - e_2) + \alpha_1 e_1 + (n-1)\alpha_n e_n =$$
$$= (-\alpha_1 - \gamma_x)e_2 + \gamma_2 e_3 + \gamma_3 e_4 + ((n-1)\alpha_n + \gamma_{n-1})e_n$$
$$D[x, e_1]_{Lie} = [D(x)e_1]_{Lie} + [x, D(e_1)]_{Lie}$$
$$D[(x, e_1) + (e_1, x)] = D(x)e_1 + e_1 D(x) + xD(e_1) + D(e_1)x$$
$$D(-e_1 - e_2 + e_1) = D(x)e_1 + e_1 D(x) + xD(e_1) + D(e_1)x$$

On the other hand, $D(e_2) = \beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_n e_n + \beta_x x$.

Therefore, we get

$$\beta_1 = 0,\ \beta_x = 0,\ \beta_2 = (\alpha_1 + \gamma_x), \beta_n = (n-1)\alpha_n + \gamma_{n-1},\ \beta_i = \gamma_{i-1},\ 3 \le i \le n-1.$$

Considering

$$D[x, x]_{Lie} = [D(x)x]_{Lie} + [xD(x)]_{Lie} = 0$$
$$D[x, x]_{Lie} = (\gamma_2 e_2 + \ldots + \gamma_n e_n)x + x(\gamma_2 e_2 + \ldots + \gamma_n e_n) = \gamma_2 e_2 + 2\gamma_3 e_3 + \ldots + (n-1)\gamma_{n-1} e_n = 0$$

we have $\gamma_2 = 0$, $\gamma_3 = 0$, ..., $\gamma_{n-1} = 0$, which implies $\beta_3 = 0$,..., $\beta_n = 0$, $\alpha_n = 0$

Now consider

$$D[e_2, x]_{Lie} = [\beta_2 e_2]x + x[\beta_2 e_2] + e_2[\gamma_1 e_1 + \gamma_n e_n + \gamma_x x] + [\gamma_1 e_1 + \gamma_n e_n + \gamma_x x]e_2 =$$
$$= (\beta_2 + \gamma_x)e_2 + (\gamma_1)e_3 = D(e_2)$$

On the other hand, $D(e_2) = \beta_2 e_2$

Therefore, we get $\gamma_x = 0$, $\gamma_1 = 0$. Thus, we have

$$D(e_1) = \alpha_1 e_1 = d(e_1)\quad D(e_2) = \beta_2 e_2 = d(e_2)$$

Applying the property of derivation for the products $[e_i, e_1] = e_{i+1}$, it is easy to get by induction that

$$D(e_i) = [(i-2)\alpha_1 + \beta_2]e_i = d(e_i),\ 2 \le i \le n$$

Therefore all Lie-derivations of the Leibniz algebra R_1 has the following form:

$$\begin{cases} D(e_1) = \alpha_1 e_1 \\ D(e_i) = [(i-2)\alpha_1 + \beta_2]e_i,\ 2 \le i \le n \end{cases}$$

Hence, $dimDerLie(R_1) = dim\,Der(R_1) = n$, thus any Lie-derivation of solvable Leibniz algebra R_1 is derivation.

Similarly, we obtain the description of Lie-derivation of the algebras R_2, R_3 and R_4 as follows:

$$Der(R_2) = DerLie(R_2) = \begin{cases} D(e_1) = \alpha_1 e_1 \\ D(e_i) = [(i-2)\alpha_1 + \beta_2]e_i \ ,2 \le i \le n \end{cases}$$

$$Der(R_3) = DerLie(R_3) = \begin{cases} D(e_1) = \alpha_1 e_1 \\ D(e_i) = [(i-2)\alpha_1 + \beta_2]e_i \ ,2 \le i \le n \end{cases}$$

$$Der(R_4) = DerLie(R_4) = \begin{cases} D(e_1) = \alpha_1 e_1 \\ D(e_i) = [(i-2)\alpha_1 + \beta_2]e_i + \beta_3 e_{i+1}, 2 \le i \le n \end{cases}$$

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